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Stability of bound states for (1 + 1)-dimensional non-linear scalar fields

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Abstract. We study the stability problem for bound states of (1 + 1)-dimensional scalar fields considering different definitions of stability. Sharp stability and instability conditions for Liapunov stability are established by improving the Shatah-Strauss formalism and a recent technique of Grillakis, Shatah and Strauss based on an analysis of the linearised operators. The linear dynamical stability is investigated for the first time and a sharp stability condition is obtained. Explicit results for the regimes of stability and instability are given for power-like self-interactions and for the ϕ^4-ϕ^6 model.

1. Introduction

In this paper we study the stability of bound states of the (classical) non-linear Klein-Gordon equation (NLKG)

$$\phi_{tt} - \phi_{xx} - g(|\phi|^2)\phi = 0. \qquad \text{(NLKG)}$$

By a bound state we mean a solution of the form

$$\phi(x, t) = \mathrm{e}^{\mathrm{i}\omega t} u_\omega(x)$$

with ω real and $u_\omega(x)$ vanishing in a suitable way as $x \to \pm\infty$.

Our aim is to investigate the stability problem for the two following definitions of stability used in the literature.

1.1. Liapunov stability

This definition of stability means that any initial datum near a bound state of the field equation gives rise to a solution which remains close to the bound state for all times with respect to a specified function space metric. Since (NLKG) is gauge invariant, i.e. $u_\omega(x)\exp[\mathrm{i}(\omega t + \theta)]$ is a solution for all $\theta \in \mathbb{R}$, we have to study the stability of this solution set ('orbit').

We obtain sharp stability and instability conditions for bound states of (NLKG) by improving the Shatah-Strauss formalism.

For space dimensions $N \geq 2$, Shatah and Strauss proved that if the action considered as a function of the frequency ω is convex (concave) the bound state of lowest energy—obtained by a minimisation problem in $H^1(\mathbb{R}^N)$—is stable (unstable) [1, 2].

These ideas were generalised and extended to abstract Hamiltonian systems in a recent paper by Grillakis *et al* [3]. They proved the stability/instability result by investigating the linear operator associated to the quadratic from given by the second (constrained) derivative of the energy functional. Let us briefly describe the main ideas.

For the stability proof the main step is to show that the bound state considered minimises the energy on the manifold of constant charge if and only if the action is a convex function of the frequency ω.

Instability is proven by construction of a functional in a neighbourhood of the 'orbit' of a bound state which is strictly monotone on the trajectories governed by the evolution equation.

Grillakis *et al* [3] showed that the local minimum property of the energy is both necessary and sufficient for stability. Thus the convexity criterion is equivalent to the energetic stability criterion for bound states of lowest energy.

In §§ 2 and 3 we improve these different versions of stability proofs for (NLKG). In § 2 we introduce the notation and recall a well known result of Berestycki and Lions [4] concerning the existence of bound states. We have that there is always a unique bound state satisfying the boundary conditions.

Then we extend the previous works of Shatah and Strauss [1, 2] to the case where

$$g(\rho) = -1 + \rho^p$$

for some $p > 0$. This extension relies on two facts.

First, the bound-state solutions satisfy a minimisation principle on $H^1(\mathbb{R})$ which is crucial for applying the techniques presented in [1, 2]. Finally, the scaling properties of these special non-linearities replace the dilations used in space dimensions $N \geqslant 2$.

In § 3 we present a detailed analysis of the linear operators related to the second constrained variation of the energy. The main steps of this part were done in our previous work [5] for the first time. We prove the applicability of the results obtained in [3] for a wide class of non-linearities. Nevertheless the analysis presented here is somewhat different.

1.2. Linear dynamical stability

A localised solution is said to be dynamically stable (in the sense of Liapunov) if small perturbations do not destroy it, i.e. one studies the behaviour of

$$\phi(x, t) = (u_\omega(x) + \eta(x, t))\, e^{i\omega t}$$

The first-order approximation leads to a linear evolution equation for $\eta(x, t)$. Now u_ω is said to be dynamically stable if $|\eta(x, t)|$ remains bounded for all t.

A precise mathematical treatment of the linearised evolution equation was given by Weinstein in the case of bound states of non-linear Schrödinger equations [6]. Our analysis parallels this method.

The main step for proving stability is to find a suitable vector space which is invariant under the time evolution and on which one can 'control' the pertubation.

In § 4 we prove that if the action considered as a function of ω is strictly convex then there exists only one unstable direction and u_ω is stable for all other pertubations η. To our knowledge this result is completely new for the Klein-Gordon equation and patches together the different definitions of stability.

Finally, in § 5 we present the following examples and study their stability properties by explicit calculations to give regimes of stability and instability:

(i) $g(\rho) = -1 + \rho^p$ $\quad p > 0$

(ii) $g(\rho) = -1 + \rho + \delta\rho^2$ $\quad \delta > 0$

(iii) $g(\rho) = -1 + \rho + \delta\rho^2$ $\quad \delta < 0.$

2. Stability of bound states: the Shatah–Strauss technique

2.1. Definitions and notations

For the stability analysis we will need several function spaces. We employ here the notation:

$$H^1(\mathbb{R}) = \left\{ u \,\middle|\, \|u\|_{H^1} \equiv \left(\int_{\mathbb{R}} |u_x|^2 \,dx + \int_{\mathbb{R}} |u|^2 \,dx \right)^{1/2} < \infty \right\}$$

$$H^1_s(\mathbb{R}) = \{ u \in H^1(\mathbb{R}) |_u \text{ is symmetric, i.e. } u(x) = u(-x) \}$$

$$L^P(\mathbb{R}) = \left\{ u \,\middle|\, \|u\|_P \equiv \left(\int_{\mathbb{R}} |u|^P \,dx \right)^{1/P} < \infty \right\}$$

$$L^P_s(\mathbb{R}) = \{ u \in L^P(\mathbb{R}) |_u \text{ is symmetric} \}.$$

The function space in which we will work is

$$X = H^1_s(\mathbb{R}) \oplus L^2_s(\mathbb{R})$$

the space of complex valued functions of $r = |x|$, which belong to $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$. An element of X is denoted by $\boldsymbol{u} = [u_1, u_2]$. X is regarded as a real Hilbert space with the inner product.

$$(\boldsymbol{u}, \boldsymbol{v})_X = \mathrm{Re} \int_{\mathbb{R}} \nabla u_1 \nabla \bar{v}_1 + u_1 \bar{v}_1 + u_2 \bar{v}_2 \,dx. \tag{2.1}$$

Let X^* be the real dual space of X. For $f \in X^*$ and $\boldsymbol{u} \in X$ the value of f at $\boldsymbol{u}$ is denoted by $\langle f, \boldsymbol{u} \rangle$; i$f$ is given by $\langle \mathrm{i}f, \boldsymbol{u} \rangle = -\langle f, \mathrm{i}\boldsymbol{u} \rangle$. We define an identification I: $X \to X^*$ as follows: $f = I(\boldsymbol{u}) \in X^*$,

$$\langle f, \boldsymbol{v} \rangle = \mathrm{Re} \int_{\mathbb{R}} u_1 \bar{v}_1 + u_2 \bar{v}_2 \,dx. \tag{2.2}$$

Furthermore we define a map $J : X \to X^*$

$$J = I \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

acting on $\boldsymbol{u}$ as a column vector.

Setting $G(\rho) \equiv \int_0^\rho g(s)\,ds$ we can express the physical quantities as follows:

$$E(\boldsymbol{u}) = \tfrac{1}{2} \int_{\mathbb{R}} |u_2|^2 + |\nabla u_1|^2 - G(|u_1|^2) \,dx \qquad \text{(energy)} \tag{2.4}$$

$$Q(\boldsymbol{u}) = \tfrac{1}{2} \langle J\boldsymbol{u}, \mathrm{i}\boldsymbol{u} \rangle = \mathrm{Im} \int_{\mathbb{R}} \bar{u}_1 u_2 \,dx \qquad \text{(charge)} \tag{2.5}$$

$$L(\boldsymbol{u}) = \tfrac{1}{2} \int_{\mathbb{R}} -|u_2|^2 + |\nabla u_1|^2 - G(|u_1|^2) \,dx \qquad \text{(action).} \tag{2.6}$$

Furthermore we need the functionals

$$K(\boldsymbol{u})=\tfrac{1}{2}\int_{\mathbb{R}}|u_2|^2+|\nabla u_1|^2+G(|u_1|^2)\,\mathrm{d}x \tag{2.7}$$

$$R(\boldsymbol{u})=\int_{\mathbb{R}}-|u_2|^2+|\nabla u_1|^2-g(|u_1|^2)|u_1|^2\,\mathrm{d}x. \tag{2.8}$$

Under suitable assumptions on G all these functionals are of class C^2 on X.

2.2. Standing waves

We now discuss the existence of standing waves and their dependence on the frequency. Let g: $\mathbb{R}\to\mathbb{R}$ be a differentiable function satisfying $g(0)=-1$. The non-linear Klein-Gordon equation (NLKG)

$$\phi_{tt}-\phi_{xx}-g(|\phi|^2)\phi=0$$

has a non-trivial standing wave solution with frequency ω provided

$$-u''=g_\omega(|u|^2)u$$

where $g_\omega(|u|^2)=g(|u|^2)+\omega^2$ has a non-trivial solution ($'$ denotes differentiation WRT x).

We require g_ω to satisfy the following conditions:

$$g_\omega(0)<0. \tag{2.9a}$$

$$\forall\zeta=\zeta(\omega)\text{ such that }G_\omega(\zeta)>0. \tag{2.9b}$$

Then we have the following theorem.

Theorem 2.1. The boundary-value problem

$$-u''=g_\omega(|u|^2)u \tag{2.10}$$

$$\lim_{X\to\pm\infty}u(x)=0$$

has a unique positive solution u_ω up to translations of the origin and this solution satisfies (after suitable translations of the origin)

(i) $u_\omega(x)=u_\omega(-x)$ $\quad x\in\mathbb{R}$

(ii) $u_\omega'(x)<0$ $\quad x>0$

(iii) $u_\omega\in C^2(\mathbb{R})$

(iv) $u_\omega(x)$ has exponential decay at infinity.

Proof. The proof relies on a simple phase plane analysis using the 'constant of motion'

$$u'^2(x)+G_\omega(|u(x)|^2)=0$$

for all $x\in\mathbb{R}$ (see [4]).

Let u_ω be a solution of (2.10) with $g(\rho)=-1+\rho^p$ for some $p>0$. Then u_ω satisfies the following extremum principles.

Theorem 2.2

(i) u_ω maximises the functional

$$T_p(u)=\frac{\|u\|_{2p+2}^2}{\|u'\|_2^2+(1-\omega^2)\|u\|_2^2} \tag{2.11}$$

on $H^1(\mathbb{R}^N)$

(ii) $\boldsymbol{u}_\omega=[u_\omega, \mathrm{i}\omega u_\omega]$ satisfies

$$L(\boldsymbol{u}_\omega)=\inf\left\{L(u,\mathrm{i}\omega u):\int_{\mathbb{R}}|u'|^2+(1-\omega^2)|u|^2\,\mathrm{d}x=\int_{\mathbb{R}}|u'_\omega|^2+(1-\omega^2)|u_\omega|^2\right\}$$

(iii)
$$\begin{aligned}L(\boldsymbol{u}_\omega)&=\inf\{L(u,\mathrm{i}\omega u):0\neq u\in H_s^1,\ R(u,\mathrm{i}\omega u)=0\}\\&=\inf\{L(u,\mathrm{i}\omega u):0\neq u\in H_s^1,\ R(u,\mathrm{i}\omega u)\leqslant 0\}.\end{aligned}$$

Proof. For part (i) we refer to Lieb [7]. Part (ii) is a consequence of (i). For (iii) we note that the set

$$\{u\in H^1,\ u\neq 0,\ R(u,\mathrm{i}\omega u)\leqslant 0\}$$

is bounded away from zero in X since

$$R(u,\mathrm{i}\omega u)\geqslant c_1\|u\|_{H^1}^2-c_2\|u\|_{H^1}^{2p+2}$$

for constants c_1, $c_2>0$. Using the lower semi-continuity of R we conclude the proof.

Proposition 2.3. Let u_ω be the solution of (2.10) obtained in theorem 2.1 and $\boldsymbol{u}_\omega=[u_\omega,\mathrm{i}\omega u_\omega]$. Then we have the following identities:

$$u_\omega'^2(x)+G_\omega(|u_\omega(x)|^2)=0 \tag{2.12}$$

$$\int_{\mathbb{R}}u_\omega'^2(x)\,\mathrm{d}x=\int_{\mathbb{R}}g_\omega(|u_\omega|^2)u_\omega^2\,\mathrm{d}x \tag{2.13}$$

$$K(\boldsymbol{u}_\omega)=R(\boldsymbol{u}_\omega)=0 \tag{2.14}$$

$$L(\boldsymbol{u}_\omega)=\int_{\mathbb{R}}u_\omega'^2(x)\,\mathrm{d}x \tag{2.15}$$

$$E(\boldsymbol{u}_\omega)=L(\boldsymbol{u}_\omega)+\omega Q(\boldsymbol{u}_\omega). \tag{2.16}$$

It can be seen that $\omega\mapsto u_\omega$ is a C^2-mapping into $H^1(\mathbb{R})$.

Therefore we have a C^2-curve $\boldsymbol{u}_\omega=[u_\omega,\mathrm{i}\omega u_\omega]$ in X which is a non-trivial solution of

$$\mathrm{D}E(\boldsymbol{u}_\omega)=\omega J\,\mathrm{i}\,\boldsymbol{u}_\omega$$

where D denotes the Fréchet derivative in X.

Now we consider the action $L(\boldsymbol{u}_\omega)$ as a function of the frequency ω and define

$$d(\omega)\equiv L(\boldsymbol{u}_\omega)=\int_{\mathbb{R}}|u'_\omega|^2\,\mathrm{d}x. \tag{2.17}$$

We shall always take $\omega>0$.

Lemma 2.4.

(i) $d(\omega)$ is a positive decreasing function of ω and

$$d'(\omega) = -Q(\boldsymbol{u}_\omega). \tag{2.18}$$

(ii) For fixed ω_0 let $\boldsymbol{u}(\lambda)$ be a C^2 curve such that $\boldsymbol{u}(0) = \boldsymbol{u}_{\omega_0}$ and $Q(\boldsymbol{u}(\lambda)) = Q(\boldsymbol{u}_{\omega_0})$. Then we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} E(\boldsymbol{u}(\lambda))\bigg|_{\lambda=0} = \langle(\mathrm{D}^2 E(\boldsymbol{u}_{\omega_0}) - \omega_0 \mathrm{D}^2 Q(\boldsymbol{u}_{\omega_0}))y_0, y_0\rangle \tag{2.19}$$

where

$$y_0 = \frac{\mathrm{d}\boldsymbol{u}(\lambda)}{\mathrm{d}\lambda}\bigg|_{\lambda=0}.$$

Proof. Obvious, see [2].

2.3. *Local minima and saddle points of the energy*

From now on we restrict ourselves to the monomial non-linearity

$$g(\rho) = -1 + \rho^p$$

for $p > 0$. Standing waves exist for $\omega \in (0, 1)$.

The extremum property of u_ω shown in proposition 2.2 is the heart of the technique presented below although some statements are also valid for general non-linearities.

Theorem 2.5. $d(\omega)$ is convex at ω_0 if and only if the energy functional E restricted to the manifold

$$M_0 \equiv \{\boldsymbol{u} \in X \mid Q'(\boldsymbol{u}) = Q(\boldsymbol{u}_{\omega_0})\}$$

has a local minimum at $\boldsymbol{u}_{\omega_0}$

Proof. For the necessity we observe that for any $\boldsymbol{u} = [u_1, u_2]$ and any ω we have the inequality

$$E(\boldsymbol{u}) \geqslant L(u_1, \mathrm{i}\omega u_1) + \omega Q(\boldsymbol{u}). \tag{2.20}$$

For $\boldsymbol{u} \in M_0$ in a small neighbourhood of $\boldsymbol{u}_{\omega_0}$ one can find a value of ω such that

$$\int_{\mathbb{R}} |\nabla u_1|^2\,\mathrm{d}x + (1-\omega^2)\int_{\mathbb{R}} |u_1|^2\,\mathrm{d}x = \int_{\mathbb{R}} |\nabla u_\omega|^2\,\mathrm{d}x + (1-\omega^2)\int_{\mathbb{R}} u_\omega^2\,\mathrm{d}x.$$

By proposition 2.2

$$\int_{\mathbb{R}} |u_\omega|^{2p+2}\,\mathrm{d}x \geqslant |u_1|^{2p+2}\,\mathrm{d}x$$

and therefore (2.20) becomes

$$E(\boldsymbol{u}) \geqslant d(\omega) + \omega Q(\boldsymbol{u}_{\omega_0}) = d(\omega) - \omega\, d'(\omega_0). \tag{2.21}$$

Since $d(\omega)$ is convex at ω_0 (2.21) implies

$$E(\boldsymbol{u}) \geqslant d(\omega_0) - \omega_0\, d'(\omega_0) = E(\boldsymbol{u}_{\omega_0}) \tag{2.22}$$

which proves that $E|_{M_0}$ has a local minimum at $\boldsymbol{u}_{\omega_0}$.

For the second part of the proof we define a curve $\omega \to \phi_\omega$ by

$$\boldsymbol{\phi}_\omega(x) = \lambda(\omega)\boldsymbol{u}_\omega(x) \tag{2.23}$$

where

$$\lambda(\omega) = \left(\frac{Q(\boldsymbol{u}_{\omega_0})}{Q(\boldsymbol{u}_\omega)}\right)^{1/2}$$

i.e. we perturb the amplitude of $\boldsymbol{u}_\omega$. Then

$$Q(u_\omega) = Q(\boldsymbol{u}_{\omega_0})$$

and

$$\begin{aligned} E(\boldsymbol{\phi}_\omega) &= L(\boldsymbol{\phi}_\omega) + \omega Q(\boldsymbol{\phi}_\omega) \\ &= -\omega d'(\omega_0) + \tfrac{1}{2}\lambda^2(\omega) \int_{\mathbb{R}} u_\omega'^2 + (1-\omega^2)u_\omega^2 \,\mathrm{d}x - \tfrac{1}{2}\int_{\mathbb{R}} \frac{\lambda(\omega)^{2p+2}}{p+1} u_\omega^{2p+2} \,\mathrm{d}x. \end{aligned}$$

By (2.13) and (2.14) we observe

$$L(\boldsymbol{u}_\omega) = \int_r u_\omega'^2 \,\mathrm{d}x = \frac{p}{p+2}(1-\omega^2)\int_{\mathbb{R}} u_\omega^2 \,\mathrm{d}x = \frac{p}{2p+2}\int_{\mathbb{R}} u_\omega^{2p+2}\,\mathrm{d}x \tag{2.24}$$

and therefore

$$E(\boldsymbol{\phi}_\omega) = -\omega\, d'(\omega_0) + \tfrac{1}{2}\left(\lambda^2 + \frac{p+2}{p}\lambda^2 - \frac{2}{p}\lambda^{2p+2}\right) d(\omega) \leqslant -\omega\, d'(\omega_0) + d(\omega). \tag{2.25}$$

Since $E|_{M_0}$ has a local minimum at $\boldsymbol{u}_{\omega_0}$ we have

$$E(\boldsymbol{\phi}_\omega) \geqslant E(\boldsymbol{u}_{\omega_0}) = -\omega_0\, d'(\omega_0) + d(\omega_0). \tag{2.26}$$

Combination of (2.25) and (2.26) yields that $d(\omega)$ is convex at ω_0.

Remark 2.6. An analogous theorem for general non-linearities will be obtained in § 3 by analysis of the linearised operator (see proposition 3.3).

2.4. *Stability and instability of bound states*

The Cauchy problem for (NLKG) can be written as follows:

$$J\frac{\mathrm{d}\boldsymbol{u}(t)}{\mathrm{d}t} = \mathrm{D}E(\boldsymbol{u}(t)) \qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in X. \tag{2.27}$$

There exists (at least) a unique weak solution of (2.27) (see, e.g., [8]) and we have two conserved quantities, the energy and the charge:

$$E(\boldsymbol{u}(t)) = E(\boldsymbol{u}_0)$$

$$Q(\boldsymbol{u}(t)) = Q(\boldsymbol{u}_0).$$

Now we state the stability/instability result for non-linearities of the form $g(\rho) = -1 + \rho^p$ for some $p > 0$.

Theorem 2.7.

(i) Let $d''(\omega_0) > 0$. Then $\{\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{u}_{\omega_0}\}$ is stable.
(ii) Let $d''(\omega_0) < 0$. Then $\{\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{u}_{\omega_0}\}$ is unstable.

The proof of this result runs along the lines of the papers by Shatah and Strauss [1, 2] using basically the extremum principle (theorem 2.2) and the energy stability criterion (theorem 2.5).

The explicit determination of $d(\omega)$ for $g(\rho)=-1+\rho^p$ is given in § 5.

3. Stability of bound states: analysis of the linearised operators

In this section we want to present a more detailed analysis of the critical points of the energy functional on M_0 by direct computation of the second derivative of the constrained functional and by a spectral analysis of the associated linear operators. Using these results we will prove the stability result for general non-linearities satisfying (2.9a, b).

Let $\boldsymbol{u}(\lambda)$ be a C^2 curve in M_0 such that $\boldsymbol{u}(0)=\boldsymbol{u}_{\omega_0}$. Then by (2.19) in lemma 2.4 we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}E(\boldsymbol{u}(\lambda))\bigg|_{\lambda=0}=\langle \mathrm{D}^2E(\boldsymbol{u}_{\omega_0})-\omega_0\mathrm{D}^2Q(\boldsymbol{u}_{\omega_0})y_0, y_0\rangle \tag{3.1}$$

where

$$y_0=\frac{\mathrm{d}\boldsymbol{u}(\lambda)}{\mathrm{d}\lambda}\bigg|_{\lambda=0}.$$

Setting $y_0=(x_1+\mathrm{i}y_1, x_2+\mathrm{i}y_2)$ we obtain

$$\langle \mathrm{D}^2E(\boldsymbol{u}_{\omega_0})-\omega_0\mathrm{D}^2Q(\boldsymbol{u}_{\omega_0})y_0, y_0\rangle$$

$$=(x_1, Hx_1)+(x_2, Lx_2)+\int_{\mathbb{R}}(x_2+\omega_0y_1)^2+(y_2-\omega_0x_1)^2\,\mathrm{d}x \tag{3.2}$$

where

$$H=-\Delta-g_{\omega_0}(u_{\omega_0}^2)-2u_{\omega_0}^2g'(u_{\omega_0}^2) \tag{3.3}$$

$$L=-\Delta-g_{\omega_0}(u_{\omega_0}^2) \tag{3.4}$$

and (,) denotes the usual inner product in L^2.

Since $\mathrm{d}Q(\boldsymbol{u}(\lambda))/\mathrm{d}\lambda=0$ the pair x_1 and y_2 has to satisfy the relation

$$(u_{\omega_0}, \omega_0x_1+y_2)=0. \tag{3.5}$$

In addition we need the 'modified linearised operator' introduced by Shatah and Strauss [2]:

$$T=-\Delta-g_{\omega_0}(u_{\omega_0}^2)-2u_{\omega_0}^2g'(u_{\omega_0})+\frac{4\omega_0^2}{(u_{\omega_0}, u_{\omega_0})}(u_{\omega_0}, \cdot)u_{\omega_0}. \tag{3.6}$$

Then it is easy to prove the following.

Proposition 3.1. Using the above notations we have

$$\frac{\mathrm{d}^2E(\boldsymbol{u}(\lambda))}{\mathrm{d}\lambda^2}\bigg|_{\lambda=0}\geqslant(x_1, Tx_1)+(x_2, Lx_2) \tag{3.7}$$

with equality if and only if $y_2=\delta u_{\omega_0}$ for some $\delta\in\mathbb{R}$ *and* $x_2+\omega_0y_1=0$.

Since we are interested in the sign of $d^2E/d\lambda^2$ at $\lambda=0$ it is enough to consider the quadratic forms defined by the self-adjoint operators T (or H) and L. We then have the following.

Proposition 3.2. (i) L is a non-negative operator on $H^1(\mathbb{R})$ and $\operatorname{Ker} L = \operatorname{span}\{u_{\omega_0}\}$.
(ii) H has exactly one negative eigenvalue and

$$\operatorname{Ker} H = \operatorname{span}\{u'_{\omega_0}\} \text{ on } H^1(\mathbb{R}).$$

Proof. Since $Lu_{\omega_0}=0$ and $u_{\omega_0}>0$, u_{ω_0} is the (non-degenerate) ground state of L which proves (i).

To prove (ii) we observe $Hu'_{\omega_0}=0$. Since u'_{ω_0} has a single node at $x=0$ we see by oscillation theory for ODE that zero is the second eigenvalue of H. The second part of the proposition is proven in [6].

By proposition 3.1 we obtain the lower bound

$$\left.\frac{d^2E(\boldsymbol{u}(\lambda))}{d\lambda^2}\right|_{\lambda=0} \geqslant (x_1, Tx_1) \tag{3.8}$$

with equality if and only if $y_2=\delta u_{\omega_0}$ for some

$$\delta\in\mathbb{R} \qquad \text{and} \qquad x_2=y_1=0 \qquad \text{or} \qquad x_2=-\omega_0 y_1 = u_{\omega_0}.$$

A detailed study of the linear operator T is given in [5]. Therefore here we only repeat the most important results.

Proposition 3.3. (i) $\alpha\equiv\inf_{(g,g)=1}(g, Tg)$ is attained for a $g^*\in L^2(\mathbb{R})$ and $\alpha=0$ iff $d''(\omega_0)\geqslant 0$.
(ii) Let $d''(\omega_0)\neq 0$. There exists a positive constant $C(\omega_0)$ such that for any $g\in L^2_s(\mathbb{R})$

$$(g, Tg)\geqslant C(\omega_0)(g,g) \qquad \text{iff } d''(\omega_0)>0. \tag{3.9}$$

As a consequence of proposition 3.3 we obtain for any C^2 curve $\boldsymbol{u}(\lambda)$ in M_0

$$\left.\frac{d^2}{d\lambda^2}E(\boldsymbol{u}(\lambda))\right|_{\lambda=0} \geqslant \tilde{C}(\omega_0)\langle y_0, y_0\rangle \tag{3.10}$$

if $d''(\omega_0)>0$.

Proposition 3.4. If $d''(\omega_0)<0$ then there exists $g_0\in L^2_s$ such that $(g_0, Tg_0)<0$.

Proof. Obvious.

Therefore we can apply the technique used by Grillakis *et al* [3] to prove the following.

Theorem 3.5. (i) Let $d''(\omega_0)>0$. Then $\{e^{i\theta}\boldsymbol{u}_{\omega_0}\}$ is stable.
(ii) Let $d''(\omega_0)<0$. Then $\{e^{i\theta}\boldsymbol{u}_{\omega_0})$ is unstable.

An easy extension [3] leads to the following.

Theorem 3.6. $\{e^{i\theta}\boldsymbol{u}_{\omega_0}\}$ is stable if and only if $d(\omega)$ is convex in a neighbourhood of ω_0.

Idea of a proof. (See [3] for details.) For the stability proof one uses (3.10). Let $\{\boldsymbol{u}_n\}$ be a sequence of initial values of problem (2.27) $\boldsymbol{u}_n(0)\to\boldsymbol{u}_{\omega_0}e^{i\theta}$ in X. Then $E(\boldsymbol{u}_n(t))\to E(\boldsymbol{u}_{\omega_0})$ by the conservation of energy. Doing a Taylor expansion of $E-\omega_0 Q$ we see that $\boldsymbol{u}_{\omega_0}$ cannot be unstable. For the proof of instability one constructs a kind of Liapunov functional using the fact that u_ω is not a local minimum of the energy E subject to constant charge Q_0.

4. The linearised evolution problem

In this section we study the linear dynamical stability of bound states. This kind of stability can be implemented if we add to $\boldsymbol{u}_\omega\, e^{i\omega t}$ a small fluctuation, i.e. we study the behaviour of

$$\phi(x,t)=(u_\omega(x)+\eta(x,t))\,e^{i\omega t} \tag{4.1}$$

as a solution of (NLKG). The first-order approximation leads to

$$\eta_{tt}+2i\omega n_t-\eta_{xx}-g_\omega(u_\omega^2)\eta-g'_\omega(u_\omega^2)u_\omega^2(\eta+\bar\eta)=0. \tag{4.2}$$

Now one calls u_ω stable if $|\eta(x,t)|$ remains bounded for all t. For one space dimension it is therefore sufficient to prove the H^1-boundness of η since H^1-boundness implies boundness in L^∞. Therefore let

$$\|\eta(x,0)\|_{H^1}+\|\eta_t(x,0)\|_{L^2}<\varepsilon.$$

Our aim is to prove the stability for almost all symmetric pertubations $[\eta,\eta_t]$ if $d(\omega)$ is convex.

Lemma 4.1. Let $[\eta,\eta_t]\in X$ be a solution of (4.2). Then the energy

$$E(\eta,\eta_t)=\tfrac12\int_{\mathbb{R}}|\eta_t|^2+|\eta_x^2|-(g_\omega(u_\omega^2)+g'(u_\omega^2)u_\omega^2)|\eta|^2\,dx$$
$$-\tfrac14\int_{\mathbb{R}}g'_\omega(u_\omega^2)u_\omega^2(\eta^2+\bar\eta^2)\,dx \tag{4.3}$$

is a conserved quantity.

Proof. Obvious.

Now it is expedient to split η into its real and imaginary parts. We set $\eta=\alpha+i\beta$. Then $[\eta,\eta_t]\in X$ is equivalent to

$$(\alpha,\alpha_t,\beta,\beta_t)^{\mathrm{T}}\in Y\equiv H_s^1\times L_s^2\times H_s^1\times L_s^2.$$

Using the definitions of the operator H and L in (3.3) and (3.4) we obtain the real system

$$\frac{d}{dt}\begin{pmatrix}\alpha\\ \alpha_t\\ \beta\\ \beta_t\end{pmatrix}=\begin{pmatrix}0&1&0&0\\ -H&0&0&2\omega\\ 0&0&0&1\\ 0&-2\omega&-L&0\end{pmatrix}\begin{pmatrix}\alpha\\ \alpha_t\\ \beta\\ \beta_t\end{pmatrix}. \tag{4.4}$$

Then the energy considered as a functional on Y is given by the following quadratic form

$$E=(\alpha_t,\alpha_t)+(\alpha,H\alpha)+(\beta_t,\beta_t)+(\beta,L\beta)\equiv Q(y) \tag{4.5}$$

where

$$y=(\alpha,\alpha_t,\beta,\beta_t)^{\mathrm{T}}\in Y.$$

We would like to show that $Q^{1/2}$ defines a norm which is equivalent to the usual norm $\|\cdot\|_Y$ on Y. Indeed this will imply the linear dynamical stability of the considered bound state.

We rewrite (4.4) as the differential equation

$$\mathrm{d}y/\mathrm{d}t=Ay \tag{4.6}$$

with initial value $y(0)=y_0\in Y$ where A denotes the matrix operator in (4.4).

Following an idea of Weinstein [6] we construct a subspace M of Y on which $Q^{1/2}$ is equivalent to the norm $\|\cdot\|_Y$ on Y.

Let

$$N_g(A)=\bigcup_{n=1}^{\infty}N(A^n)$$

be the generalised nullspace of A. We set

$$M\equiv Y\cap[N_g(A^*)]^{\perp} \tag{4.7}$$

where $\perp$ denotes orthogonality with respect to the inner product of Y and A^* is the adjoint of A.

Let us first determine the elements of $N_g(A)$ and $N_g(A^*)$.

Proposition 4.2. Let $d''(\omega)\neq 0$. We have $N_g(A)=N(A)\cup N(A^2)$ and $N_g(A^*)=N(A^*)\cup N(A^{*2})$. $N_g(A)$ and $N_g(A^*)$ are spanned by the following two-dimensional biorthogonal sets.

$$e_1=(0,0,u_\omega,0)^{\mathrm{T}} \tag{4.8a}$$

$$e_2=(-2\omega H^{-1}u_\omega,0,0,-u_\omega)^{\mathrm{T}} \tag{4.8b}$$

$$f_1=(0,2\omega H^{-1}u_\omega,-(1+4\omega^2H^{-1})u_\omega,0)^{\mathrm{T}} \tag{4.9a}$$

$$f_2=(2\omega u_\omega,0,0,u_\omega)^{\mathrm{T}} \tag{4.9b}$$

where

$$\langle e_i,f_j\rangle_Y=d''(\omega)\delta_{ij}. \tag{4.10}$$

Proof. Let $W=(w_1,w_2,w_3,w_4)^{\mathrm{T}}\in Y$, then from

$$0=AW=\begin{pmatrix} w_2\\ -Hw_1+2\omega w_4\\ w_4\\ -2\omega w_4-Lw_3\end{pmatrix}$$

it follows obviously that $w_2 = w_4 = 0$. Since H is invertible on the space of symmetric functions we have also $w_1 = 0$. $Lw_3 = 0$ implies $w_3 \sim u_\omega$. Therefore $N(A)$ is spanned by e_1, i.e. $N(A) = \langle e_1 \rangle$. If $0 = A^2 W$ we obtain the system

$$0 = -Hw_1 + 2\omega w_4 \tag{4.11a}$$

$$0 = -(H + 4\omega^2)w_2 - 2\omega L w_3 \tag{4.11b}$$

$$0 = -2\omega w_2 - Lw_3 \tag{4.11c}$$

$$0 = 2\omega H w_1 - (L + 4\omega^2) w_4. \tag{4.11d}$$

Equations (4.11a, d) imply $Lw_4 = 0$, (4.11b, c) yield $w_2 = 0$ and $Lw_3 = 0$ which implies that $N(A^2)$ is spanned by e_1 and e_2, i.e. $N(A^2) = \langle e_1, e_2 \rangle$. $A^3 W = 0$ yields

$$0 = -(H + 4\omega^2)w_2 - 2\omega L w_3 \tag{4.12a}$$

$$0 = H(H + 4\omega^2)w_1 - 2\omega(H + 4\omega^2 + L)w_4 \tag{4.12b}$$

$$0 = -(L + 4\omega^3)w_4 + 2\omega H w_1 \tag{4.12c}$$

$$0 = L(L + 4\omega^3)w_3 + 2\omega(H + 4\omega^2 + L)w_2. \tag{4.12d}$$

Equations (4.12b, c) imply

$$(H + 4\omega^2)(L + 4\omega^2)w_4 = 4\omega^2(H + 4\omega^2 + L)w_4.$$

Therefore $HLw_4 = 0$ which yields $w_4 \sim u_\omega$ and $2\omega H^{-1} w_4 = w_1$. Equations (4.12a, d) give

$$(L + 4\omega^2)(H + 4\omega^2)w_2 = 4\omega^2(H + 4\omega^2 + L)w_2$$

i.e. $LHw_2 = 0$. But then

$$2\omega L w_3 = \lambda(1 + 4\omega^2 H^{-1})u_\omega.$$

Taking the L^2 product with u_ω we obtain $0 = -\lambda d''(\omega)$ (see [5]), which implies $\lambda = 0$ or $w_2 = 0$. Then $w_3 \sim u_\omega$. Therefore $N(A^3)$ is spanned by e_1 and e_2 and $N(A^3) = N(A^2)$. An easy consequence is $N(A^n) = N(A^2)$ for all $n \geq 3$. Since H and L are self-adjoint we have $A^* = A^{\mathrm{T}}$ where A^{T} denotes the linear operator associated to the transposed matrix. Since

$$A^* W = \begin{pmatrix} -Hw_2 \\ w_1 - 2\omega w_4 \\ -Lw_4 \\ 2\omega w_2 + w_3 \end{pmatrix}$$

we see immediately that $N(A^*)$ is spanned by f_2. The equation $A^{*2} W = 0$ leads to the system

$$0 = -Hw_1 + 2\omega H w_4 \tag{4.13a}$$

$$0 = -(H + 4\omega^2)w_2 - 2\omega w_3 \tag{4.13b}$$

$$0 = -2\omega L w_2 - Lw_3 \tag{4.13c}$$

$$0 = 2\omega w_1 - (4\omega^2 + L)w_4. \tag{4.13d}$$

By (4.13a) we see $w_1 = 2\omega w_4$. Thus (4.13d) implies $Lw_4 = 0$. By (4.13c) we obtain $2\omega w_2 + w_3 = \lambda u_\omega$ which yields $-Hw_2 = 2\omega\lambda u_\omega$ by (4.13b). Therefore $N(A^{*2})$ is spanned by f_1 and f_2.

Finally, computing $A^{*3}w=0$, we obtain

$$0=H(H+4\omega^2)w_2+2\omega Hw_3 \tag{4.14a}$$

$$0=-(H+4\omega^2)w_1+2\omega(H+4\omega^2+L)w_4 \tag{4.14b}$$

$$0=-2\omega Lw_1+L(L+4\omega^2)w_4 \tag{4.14c}$$

$$0=-(L+4\omega^2)w_3-2\omega(H+4\omega^2+L)w_2. \tag{4.14d}$$

Equations (4.14*a*, *d*) imply

$$(L+4\omega^2)(H+4\omega^2)w_2=4\omega^2(H+4\omega^2+L)w_2.$$

Therefore $w_2=\lambda H^{-1}u_\omega$. On the other hand (4.14*a*, *d*) yield $L(w_3+2\omega w_2)=0$. Therefore these elements of $N(A^{*3})$ are generated by f_1. By equation (4.14*c*) we have

$$2\omega w_1=(L+4\omega^2)w_4-\lambda u_\omega.$$

Inserting this into (4.14*b*) we obtain $0=-HLw_4+\lambda(H+4\omega^2)u_\omega$. Using the invertibility of H and taking the L^2 product with u_ω we obtain $0=-\lambda d''(\omega)$. Therefore $\lambda=0$ and $Lw_4=0$. We see that these elements are generated by f_2. Therefore $N(A^{*2})=N(A^{*3})=N(A^{*n})$. The orthogonality relations (4.10) are obvious. Thus proposition 4.2 is proved.

By the biorthogonality of $N_g(A)$ and $N_g(A^*)$ we have

$$Y\simeq M\oplus N_g(A). \tag{4.15}$$

The stability of $N_g(A)$ WRT to the evolution of the system (4.6) is described in the following proposition.

Proposition 4.3. Let $W(t)$ be a solution of (4.6) with $W(0)\in N_g(A)$. Then $W(t)\in N_g(A)$ for all t and

$$W(t)=d''(\omega)^{-1}[(W(t),f_1)e_1+(W(t),f_2)e_2] \tag{4.16}$$

and especially

$$(W(t),f_2)=(W(0),f_2) \tag{4.17a}$$

$$(W(t),f_1)=(W(0),f_2)t+(W(0),f_1) \tag{4.17b}$$

where (,) denotes the usual inner product of $(L^2)^4$.

Proof. The representation (4.16) is clear from the biorthogonality of the sets $\{e_1, e_2\}$ and $\{f_2, f_2\}$.

Now let $W(t)=c_1(t)e_1+c_2(t)e_2$. Inserting this expression into (4.6) yields

$$\dot{c}_1(t)e_1+\dot{c}_2(t)e_2=c_2(t)e_1$$

which implies

$$c_2(t)=c_2(0) \qquad \text{and} \qquad c_1(t)=c_2(0)t+c_1(0).$$

This is precisely (4.17*a*, *b*).

Thus if $W(0)$ has a vanishing component in f_2, then so will $W(t)$. As a consequence of proposition 4.3 we have the following corollary.

Corollary 4.4. M is an invariant subspace for $\Omega(t)=\exp(tA)$.

Next we show that the restriction on M of the quadratic form Q defines a norm which is equivalent to $\|\cdot\|_y$ on M.

Theorem 4.5. Let $y=(\alpha, \alpha_t, \beta, \beta_t)^{\mathrm{T}} \in M$. If $d''(\omega)>0$, then there exist positive constants K_1 and K_2 such that

$$K_1 \| y \|_Y^2 \leqslant Q(y) \leqslant K_2 \| y \|_Y^2 \tag{4.18}$$

or equivalently

$$\begin{aligned} K_1(\| \alpha \|_{H^1}^2 + \| \alpha_t \|_2^2 + \| \beta \|_{H^1}^2 + \| \beta_t \|_2^2) & \\ \leqslant E(\alpha, \alpha_t, \beta, \beta_t) & \\ \leqslant K_2(\| \alpha \|_{H_1}^2 + \| \alpha_t \|_2^2 + \| \beta \|_{H^1}^2 + | \| \beta_t \|_2^2). & \end{aligned} \tag{4.19}$$

Proof. The proof of the existence of an upper bound is easy and it holds for any $y \in Y$. To prove the lower estimate we have to use the orthogonality relations $(y, f_1) = (y, f_2) = 0$ which hold for any $y \in M$. They can be written as follows:

$$(\alpha_t, 2\omega H^{-1} u_\omega) - (\beta, (1+4\omega^2 H^{-1}) u_\omega) = 0 \tag{4.20a}$$

$$(\alpha, 2\omega u_\omega) + (\beta_t, u_\omega) = 0. \tag{4.20b}$$

From the results of § 3 we know that H and L are self-adjoint operators satisfying

$$(\alpha, H\alpha) \geqslant -C \| \alpha \|_{H^1}^2 \tag{4.21a}$$

$$(\beta, L\beta) \geqslant 0 \tag{4.21b}$$

for any α, $\beta \in H^1(\mathbb{R})$ and some $C>0$.

By the orthogonality relations (4.20*a*, *b*) we have

$$(\alpha_t, \alpha_t) + (\beta, L\beta) \geqslant (\beta, S\beta) \tag{4.22a}$$

$$(\beta_t, \beta_t) + (\alpha, H\alpha) \geqslant (\alpha, T\alpha) \tag{4.22b}$$

where S and T are linear operators defined by

$$S = L + \frac{((1+4\omega^2 H^{-1}) u_\omega, \cdot)}{4\omega^2 (H^{-1} u_\omega, H^{-1} u_\omega)} (1+4\omega^2 H^{-1}) u_\omega \tag{4.23a}$$

$$T = H + 4\omega^2 \frac{(u_\omega, \cdot)}{(u_\omega, u_\omega)} u_\omega. \tag{4.23b}$$

In § 3 (or [5]) it is shown that if $d''(\omega)>0$ then

$$(\alpha, T\alpha) \geqslant C(\omega) \| \alpha \|_{H^1}^2 \tag{4.24}$$

holds for a positive constant $C(\omega)$.

If $d''(\omega)>0$ by using similar methods one can also prove that for all $\beta \in H^1$ we have

$$(\beta, S\beta) \geqslant \tilde{C}(\omega) \| \beta \|_{H^1}^2 \tag{4.25}$$

for a positive constant $\tilde{C}(\omega)$.

The idea leading to proof of (4.25) can be stated as follows. Obviously S is non-negative. One shows as in [5] that if the infimum on the unit sphere of $(\beta, S\beta)$ is zero then it is attained by some $\beta^* \in H^1$. Applying the Lagrange multiplier theorem we obtain $S\beta^* = 0$. Then we cannot have $(\beta^*, (1+4\omega^2 H^{-1}) u_\omega) = 0$ (by contradiction). But taking the inner product with u_ω will lead to $d''(\omega) = 0$ which is impossible. Thus we have a non-zero infimum which implies (4.25).

Applying (4.21)-(4.25) yields

$$(\alpha_t, \alpha_t)+(\beta_t, \beta_t)+(\alpha, H\alpha)+(\beta, L\beta)$$
$$\geqslant [(1-\varepsilon)C(\omega)-\varepsilon C]\|\alpha\|_{H^1}^2+\varepsilon\|\alpha_t\|_2^2+\tfrac{1}{2}\|\beta_t\|_2^2+\tfrac{1}{2}C(\omega)\|\beta\|_{H^1}^2.$$

Choosing ε small enough we obtain the desired lower estimate.

Now we are in a position to prove our stability result.

Theorem 4.6. Let $d''(\omega)>0$. Then u_ω is stable in the sense of linear dynamical stability with respect to all pertubations (η, η_t) such that $(\mathrm{Re}\,\eta, \mathrm{Re}\,\eta_t, \mathrm{Im}\,\eta, \mathrm{Im}\,\eta_t)\in M$.

Proof. Since M is invariant under the linear evolution equation by corollary 4.4. the pertubation will stay for all times in M.

Thus we have an H^1-control by the conservation of the energy (theorem 4.5 and lemma 4.1), i.e.

$$\|\eta(x,t)\|_{H^1}^2+\|\eta_t(x,t)\|_{L^2}^2<K'E(\eta,\eta_t)<K(\|\eta(x,0)\|_{H^1}^2+\|\eta_t(x,0)\|_{L^2}^2). \tag{4.26}$$

Since $\eta(x,t)$ is bounded in $H^1(\mathbb{R})$ for all t, $\eta(x,t)$ is bounded in L^∞ (it is even Hölder continuous with exponent $\frac{1}{2}$) for all t by the inequality

$$|\eta(x,t)|^2\leqslant\|\eta(x,t)\|_{H^1}^2\leqslant C\|\eta(x,0)\|_{H^1}^2 \tag{4.27}$$

which proves the theorem.

Finally we want to explain why it is rather natural that M is the space of 'stable pertubations'.

For this purpose we consider the elements of the complementing space $N_g(A)$ in Y. For the elements e_1 and e_2 we have

$$Q(e_1)=0 \tag{4.28a}$$

$$Q(e_2)=4\omega^2(u_\omega, H^{-1}u_\omega)+(u_\omega, u_\omega)$$
$$=-d''(\omega)<0 \qquad \text{if } d''(\omega)>0. \tag{4.28b}$$

Perturbations proportional to e_1 remain constant while perturbations in the direction of e_2 grow linearly in time.

These properties are closely related to the spectrum of the linearised operator of the problem defined in (3.1) and (3.2).

It is easy to see that zero belongs to the spectrum of the linearised operator which comes from the gauge invariance of the non-linear problem. If $d''(\omega)>0$ then the linearised operator also has some negative spectrum (see also [3]).

This fact is expressed for the linear problem by (4.28*a*, *b*).

Physically this situation may be interpreted as follows. The functions

$$\psi(x,t,\theta,\Omega)=u_\Omega(x)\,\mathrm{e}^{\mathrm{i}(\Omega t+\theta)} \tag{4.29}$$

form a two-parameter family of bound states for (NLKG). It is easy to check that the time evolution of elements in $N_g(A)$ described in proposition 4.3 corresponds to the derivatives of ψ with respect to the free parameters θ and Ω at $\theta=0$ and $\Omega=\omega$. Indeed

$$\frac{\partial\psi}{\partial\theta} \leftrightarrow e_1$$

$$\frac{\partial\psi}{\partial\Omega} \leftrightarrow e_2 + te_1$$

where e_1 is a tangent at the orbit of u_ω. Since the linearisation (4.1) considers the perturbation in the 'rest frame' of the solitary wave $u_\omega \exp(\mathrm{i}\omega t + \mathrm{i}\theta)$, solutions of the linearised equation (4.2) in the direction of e_1 remain constant. The derivative of ψ with respect to Ω corresponds to the first-order approximation of the motion of nearby solitary waves with different frequency:

$$\psi(\cdot,\Omega) = \psi(\cdot,\omega) + (\Omega-\omega)\left.\frac{\partial\psi}{\partial\Omega}\right|_{\Omega=\omega} + \mathrm{O}[(\Omega-\omega)^2].$$

To summarise, the time evolution of the elements in $N_g(A)$ does not come from physical properties of the solitary wave itself but from the linearisation procedure. Hence physically meaningful perturbations are only in M. In particular, if $d''(\omega)>0$ then zero is a local minimum of the linearised energy in M and the unstable direction is covered by the secular modes e_1 and e_2.

5. Examples

In this section we present a number of examples which arise from the applications (e.g. [9]).

5.1. $g(\rho) = -1 + \rho^p,\ p>0$

The equation

$$\phi_{tt} - \phi_{xx} + \phi - |\phi|^{2p}\phi = 0 \tag{5.1}$$

admits standing waves $\phi(x,t) = \mathrm{e}^{\mathrm{i}\omega t}u_\omega(x)$ by theorem 2.1 if and only if $0 \leqslant \omega^2 < 1$. u_ω satisfies the stationary equation

$$-u''_\omega = -(1-\omega^2)u_\omega + u_\omega^{2p+1}. \tag{5.2}$$

We compute $d(\omega)$ explicitly. Note that

$$v(x) = \lambda^{-1/p}u_\omega(x/\lambda) \qquad \lambda = (1-\omega^2)^{1/2}$$

satisfies the non-linear ordinary differential equation

$$-v'' = -v + v^{2p+1}$$

and thus

$$\mathrm{d}(\omega) = (1-\omega^2)^{1/p+1/2}d(0).$$

Calculating the second derivative of $d(\omega)$ we obtain

$$d''(\omega) = \left(\frac{2}{p}+1\right)\left(\frac{2}{p}\omega^2-1\right)(1-\omega^2)^{1/p-3/2}d(0).$$

Applying theorem 2.15 we have the following proposition.

Proposition 5.1 (i) If $p \geq 2$ all standing waves are unstable.

(ii) If $p \leq 2$ the standing waves are unstable if $|\omega| \leq \omega_C$; the standing waves are stable if $\omega_C < |\omega| < 1$, where $\omega_C = (p/2)^{1/2}$.

Because of the scaling properties of (5.2) we were not forced to calculate the explicit representation of u_ω when studying the stability problem.

The knowledge of the explicit solutions becomes more relevant for detailed studies of the equation and the linearised problem.

For given p the solution u_ω of (5.2) has the form

$$u_\omega(x) = (p+1)^{1/2p}(1-\omega^2)^{1/2p} \cosh^{-1/p} p(1-\omega^2)^{1/2}x \tag{5.3}$$

from which one can calculate directly (the ingredients) of the linearised analysis.

5.2. $g(\rho) = -1 + \rho + \delta\rho^2$, $\delta > 0$

We consider the non-linear Klein-Gordon equation

$$\phi_{tt} - \phi_{xx} + \phi - |\phi|^2\phi - \delta|\phi|^4\phi = 0 \tag{5.4}$$

to which corresponds to the standing wave equation given by

$$-u''_\omega = -(1-\omega^2)u_\omega + u_\omega^3 + \delta u_\omega^5 \tag{5.5}$$

for $0 \leq \omega^2 < 1$. We rescale (5.5) by setting

$$w_\omega(x) = \lambda^{-1}u_\omega(x/\lambda) \qquad \lambda = (1-\omega^2)^{1/2}. \tag{5.6}$$

Then w_ω is a solution of the equation

$$-w''_\omega = -w_\omega + w_\omega^3 + (1-\omega^2)\delta w_\omega^5. \tag{5.7}$$

In order to compute $d(\omega)$ explicitly we have to know the explicit form of w_ω. By integrating (5.7) or by inspection we see that w_ω has the representation

$$w_\omega(x) = 2(1 + b(\omega)\cosh 2x)^{-1/2} \tag{5.8}$$

where $b(\omega) = [1 + \frac{16}{3}\delta(1-\omega^2)]^{1/2}$.

Next we calculate the charge Q of u_ω and differentiate with respect to ω. We obtain

$$Q(\boldsymbol{u}_\omega) = \omega(1-\omega^2)^{1/2}\| w_\omega \|_2^2. \tag{5.9}$$

Using

$$\| w_\omega \|_2^2 = 4(\pi/2)^{1/2} b^{-1/2}(\omega)(b^2(\omega)-1)^{-1/4} P_{-1/2}^{-1/2}(b^{-1}(\omega)) \tag{5.10}$$

where

$$P_{-1/2}^{-1/2}(\cos\phi) = \left(\frac{2}{\pi \sin\phi}\right)^{1/2} \phi$$

is the associated Legendre function of the first kind we finally obtain

$$Q(\boldsymbol{u}_\omega) = \left(\frac{3}{\delta}\right)\omega \cos^{-1}\frac{1}{b(\omega)}. \tag{5.11}$$

Now we state our stability result.

Proposition 5.2. There exists $\omega_c > 0$ such that

(i) $d''(\omega) < 0$ if $0 \leqslant \omega < \omega_c$

(ii) $d''(\omega) > 0$ if $\omega > \omega_c$.

As a consequence u_ω is unstable if $0 \leqslant \omega \leqslant \omega_c$ and is stable if $\omega_c < \omega < 1$.

Proof. We compute

$$\frac{\mathrm{d}Q(\boldsymbol{u}_\omega)}{\mathrm{d}\omega} = \left(\frac{3}{\delta}\right)^{1/2} \cos^{-1}\frac{1}{b(\omega)} + \left(\frac{3}{\delta}\right)^{1/2} \omega \frac{\mathrm{d}b^{-1}(\omega)}{\mathrm{d}\omega}\left(-\frac{1}{(1-b^{-2}(\omega))^{1/2}}\right)$$

$$= \left(\frac{3}{\delta}\right)^{1/2}\left(\cos^{-1}\frac{1}{b(\omega)} - \frac{16}{3}\delta\omega^2\frac{1}{b^3(\omega)}\frac{1}{(1-b^{-2}(\omega))^{1/2}}\right).$$

Setting $1/b(\omega) = \cos\phi(\omega)$ we have

$$\frac{\mathrm{d}Q(\boldsymbol{u}_\omega)}{\mathrm{d}\omega} = \left(\frac{3}{\delta}\right)^{1/2} \cot\phi(\phi\tan\phi + 1 - b^2(0)\cos^2\phi). \tag{5.12}$$

The right-hand side of (5.12) is strictly increasing in ϕ. It is negative for small ϕ and positive for ϕ close to $\cos^{-1}(1/b(0))$. This proves the proposition where ω_c is determined by the equation

$$\phi(\omega_c)\tan\phi(\omega_c) + 1 - b^2(0)\cos^2\phi(\omega_c) = 0. \tag{5.13}$$

5.3. $g(\rho) = -1 + \rho + \delta\rho^2$, $\delta < 0$

Again we consider the non-linear Klein-Gordon equation given by (5.4). It is easy to see that standing waves exist only in the following range of frequencies

$$\begin{aligned} &1 > \omega^2 > \omega^{*2} = 1 + \frac{3}{16\delta} && \text{if } \delta \leqslant -\tfrac{3}{16} \\ &1 > \omega^2 \geqslant 0 && \text{if } \delta > -\tfrac{3}{16}. \end{aligned} \tag{5.14}$$

Again w_ω defined by (5.6) is of the form

$$w_\omega(x) = 2(1 + b(\omega)\cosh 2x)^{-1/2} \tag{5.15}$$

where $b(\omega) = [1 + \frac{16}{3}\delta(1-\omega^2)]^{-1/2}$. Note that since $\delta < 0$ we now have $b(\omega) < 1$.

We compute

$$\| w_\omega \|_2^2 = \frac{4}{(1-b^2(\omega))^{1/2}} \cosh^{-1}\frac{1}{b(\omega)} \tag{5.16}$$

which yields

$$Q(\boldsymbol{u}_\omega) = \left(\frac{3}{-\delta}\right)^{1/2} \omega \cosh^{-1}\frac{1}{b(\omega)}. \tag{5.17}$$

Using the above expression for the charge it is easy to obtain the following result about the stability of u_ω.

Proposition 5.3. (i) Let $\delta \geqslant -\frac{3}{16}$. Then there exists $\omega_c > 0$ such that u_ω is stable for $\omega > \omega_c$ and unstable for $\omega \leqslant \omega_c$.

(ii) Let $\delta < -\frac{3}{16}$. There exists $\delta_0 < -\frac{3}{16}$ such that all standing waves are stable if $\delta < \delta_0$. If $\delta = \delta_0$ all standing waves are stable since $d(\omega)$ is convex everywhere. If $\delta_0 < \delta < -\frac{3}{16}$ there exists a closed interval of unstable frequencies.

Proof. As in the proof of proposition 5.2 we obtain

$$\frac{\mathrm{d}Q(\boldsymbol{u}_\omega)}{\mathrm{d}\omega} = \left(\frac{3}{-\delta}\right)^{1/2} \coth\psi(\psi \tanh\psi + \sinh^2\psi + \tfrac{16}{3}\delta\cosh^2\psi) \tag{5.18}$$

where

$$\psi = \psi(\omega) = \cosh^{-1}\frac{1}{b(\omega)}.$$

Consider now $f(\psi) = \psi\tanh\psi - 1 + (1 + \frac{16}{3}\delta)\cosh^2\psi$.

If $\delta > -\frac{3}{16}$ we have

$$f(\psi) < 0 \qquad \text{as } \psi \to 0 \qquad \text{i.e. } \omega \to 1$$

$$f(\psi) > 0 \qquad \text{as } \psi \to \cosh^{-1}\frac{1}{b(0)} \qquad \text{i.e. } \omega \to 0.$$

If $\delta = -\frac{3}{16}$ we have $f(\psi) = \psi\tanh\psi - 1$ and we conclude

$$f(\psi) < 0 \qquad \text{as } \psi \to 0 \qquad \text{i.e. } \omega \to 1$$

$$f(\psi) > 0 \qquad \text{as } \psi \to \infty \qquad \text{i.e. } \omega \to 0.$$

If $\delta < -\frac{3}{16}$ then $1 + \frac{3}{16}\delta < 0$ therefore

$$f(\psi) < 0 \qquad \text{as } \psi \to 0 \qquad \text{i.e. } \omega \to 1$$

$$f(\psi) < 0 \qquad \text{as } \psi \to \infty \qquad \text{i.e. } \omega \to \omega^*.$$

For δ large enough, $f(\psi)$ is everywhere negative. Otherwise there exists an interval on which $f(\psi)$ is positive. Thus we prove proposition 5.3.

We compute δ_0 numerically. δ_0 is determined by the equations $f(\psi_0) = 0$ and $f'(\psi_0) = 0$, i.e.

$$\begin{aligned} &-\psi_0\tanh\psi_0 + 1 - 1(1 + \tfrac{16}{3}\delta_0)\cosh^2\psi_0 = 0 \\ &-\tanh\psi_0 - \frac{\psi_0}{\cosh^2\psi_0} - 2(1 + \tfrac{16}{3}\delta_0)\sinh\psi_0\cosh\psi_0 = 0. \end{aligned} \tag{5.19}$$

Eliminating δ_0 leads to the equation

$$2\psi_0\sinh\psi_0 - 3\sinh\psi_0\cosh\psi_0 - \psi_0 = 0.$$

Defining $x_0 = 2\psi_0$ we obtain by simple transformations

$$x_0\cosh x_0 - 2x_0 - 3\sinh x_0 = 0$$

which we solve numerically using Newton's method. This yields

$$\psi_0 = 1.717\,92\ldots.$$

Using this value we compute

$$\delta_0 = -0.201\,34\ldots$$

and

$$\omega_0 = 0.181\,325\ldots.$$

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